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CRITERIA FOR NODES IN DUPIN'S CYCLIDES, WITH A CORRESPONDING CLASSIFICATION.*

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Since the introduction of generalized coördinates, new surfaces have engaged the attention of mathematicians, and known ones have been found to possess many new and interesting properties. Prominent among these are the cyclides, surfaces of the fourth order defined as the envelope of a system of spheres, fulfilling either two or three conditions, and especially Dupin's cyclide, defined as the envelope of a singly infinite system of spheres which touch three given ones.

In the classification of cyclides by Loria† the cyclide of Dupin only appears as a single type: the specific forms of these surfaces are not considered. Loria uses five spheres to define his system of (point) coördinates, discussed as "coördinées pentasphériques" by Darboux‡ and the general cyclide is expressed by a quadratic equation.

By using the hexaspherical coördinates introduced by Lie§ the sphere, not the point, is regarded as the generating element. There is a close analogy between this geometry and the geometry of the straight line as introduced by Plücker,|| Dupin's cyclide playing exactly the same rôle in the former, as the hyperboloid of one nappe in the latter. In a recent paper¶ the writer gave the details of this analogy for the linear equation. An imaginary transformation is there employed which makes it impossible to determine criteria for reality of any particular properties.

In this paper it is proposed to derive the criteria for nodes in Dupin's Cyclides, using the symbols and nomenclature employed in the previous paper, but without making any further reference to line geometry.

It will be sufficient to give an outline of the notation and terms used.

Let

$$(\mu - \nu)(x^2 + y^2 + z^2) - 2\xi xt - 2\eta yt - 2\zeta zt + (\mu + \nu)t^2 = 0 \quad (1)$$

* Read at the Buffalo meeting of the American Mathematical Society.

† G. LORIA: *Recerche intorno alla geometria della sfera e loro applicazione*, . . . *Memoire della Accademia Reale di Torino*, II, 36, 1884.

‡ G. DARBOUX: *Sur une Classe remarquable de Courbes et de Surfaces* . . . Paris, 1873. 2d Edition, 1896.

§ S. LIE: *Ueber Complexe, insbesondere über Linieri eine Kugelcomplexe und ihre Anwendung auf die Theorie der Differentialgleichungen*. Math. Ann. BV, 1872.

|| J. PLÜCKER: *Neue Geometrie des Raumes, mit der geraden Linie als Raum element*. Leipzig, 1868-9.

¶ Ueber die linearen Complexe der Lie'schen Kugelgeometrie. Göttingen, 1895.

be the homogeneous equation of a sphere; denote the radius by $\frac{\lambda}{\mu - \nu}$, then the quadratic relation

$$H - \xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \mu^2 + \nu^2 = 0 \quad (2)$$

exists between these quantities, which may be called the *six homogeneous coördinates of the sphere*; conversely, any six quantities which fulfil this relation, are sufficient to fix a sphere.

Two spheres ξ, η, ζ ; $\xi', \eta', \zeta' \dots$ cut each other at an angle φ defined by the equation

$$\xi\xi' + \eta\eta' + \zeta\zeta' - \lambda\lambda' \cos \varphi - \mu\mu' + \nu\nu' = 0. \quad (3)$$

Now consider a general linear equation with real coefficients, between $\xi, \eta, \dots$,

$$H - a\xi + b\eta + c\zeta + d\lambda + e\mu + f\nu = 0. \quad (4)$$

The sphere whose coördinates satisfy this equation constitute a *linear spherical complex*. Make (4) identical with (3); by putting

$$a = \xi'; b = \eta'; c = \zeta', d = -\lambda' \cos \varphi; e = -\mu', f = \nu'.$$

This can always be done for the only condition required of $\xi', \eta', \dots$ is $H = 0$ and $\cos \varphi$ is assignable; hence

A *linear spherical complex consists of the ∞^3 spheres which cut a fixed sphere at a constant angle.*

The fixed sphere is called the *fundamental sphere*, and the constant angle, the *angle of the complex*.

The radius of the fundamental sphere, ρ , is defined by the equation

$$\rho = \frac{\pm \sqrt{a^2 + b^2 + c^2 - e^2 + f^2}}{e + f}, \quad (5)$$

and the angle of the complex, φ , by the equation

$$\cos \varphi = \frac{d}{\sqrt{a^2 + b^2 + c^2 - e^2 + f^2}}. \quad (6)$$

The expression

$$S \equiv a^2 + b^2 + c^2 - d^2 - e^2 + f^2 \quad (7)$$

is called the *invariant* of the complex. When $\varphi = 0$, in (6) $S = 0$, and the complex becomes *special*; it consists of all the spheres which touch the sphere (a, b, c, d, e, f) .

A point may be regarded as a sphere with a vanishing radius (point sphere), and a plane as a sphere with an infinite radius. There are ∞^3 spheres

having a vanishing radius, which is expressed by the condition $\lambda = 0$; there are ∞^3 spheres having an infinite radius, which is expressed by the condition $\mu - \nu = 0$ [cf. eq. (1)], hence

The points and planes of space form each a linear spherical complex; their equations are, respectively,

$$\lambda = 0 \text{ and } \mu - \nu = 0. \quad (8)$$

The spheres common to two complexes, $\phi_1 = 0$, $\phi_2 = 0$ form a *linear spherical congruence*. The congruence evidently belongs to every complex of the pencil

$$\phi_1 + x\phi_2 = 0 \quad (9)$$

where x is a varying parameter. Among these complexes are two special ones, defined by the equation

$$S_{11} + 2xS_{12} + x^2S_{22} = 0, \quad (10)$$

where S_{11} , S_{22} are the invariants of ϕ_1 , ϕ_2 , and

$$S_{12} = a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2 - e_1e_2 + f_1f_2$$

is the *simultaneous invariant* of ϕ_1 , ϕ_2 .

The fundamental spheres of the two special complexes contained in the pencil (9) are real, coincident or imaginary, according as

$$\begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} [S_{1x} = S_{x1}] \quad (11)$$

is negative, zero, or positive.

The spheres belonging to three complexes ϕ_1 , ϕ_2 , ϕ_3 form a linear spherical *series*; they evidently belong to every complex of the sheaf

$$\phi_1 + x\phi_2 + \tau\phi_3 = 0, \quad (12)$$

x , τ being varying parameters. In this sheaf are ∞^1 special complexes, determined by those values of x , τ which satisfy the equation

$$S_{11} + 2xS_{12} + x^2S_{22} + 2x\tau S_{23} + 2\tau S_{13} + \tau^2S_{33} = 0. \quad (13)$$

The fundamental spheres of these special complexes are the *directrices* of the series. Every sphere of the series touches all the directrices, and conversely, but no two spheres of the same system can touch each other. This

configuration is a *Dupin's cyclide*; the centres of the spheres belonging to the series lie on one focal line, and those of the directrices on the other. These focal lines are conics in planes perpendicular to each other (planes of symmetry of the surface) the vertices of each being the foci of the other.*

The discriminant of the invariant (13)

$$\Delta \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix}, \quad (14)$$

being of odd order, has no importance attached to its sign. When $\Delta = 0$, the cyclide reduces to a gauche quadrilateral of absolute lines.†

There is a finite number of spheres belonging to four complexes, $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = 0$; there being four linear and one quadratic equation (2), defining $\xi, \eta, \dots$ the number is seen to be two.‡

These spheres may be real or imaginary; they belong to the ∞^3 complexes of the group

$$\phi_1 + \alpha\phi_2 + \tau\phi_3 + \sigma\phi_4 = 0. \quad (15)$$

The invariant of this complex is a function of α, τ, σ . The complex becomes special for values of α, τ, σ which lie on a quadric surface, whose discriminant is

$$D \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix}. \quad (16)$$

Analogous to Eq. (11), the two spheres common to four complexes are real, coincident or imaginary, according as D is negative, zero, or positive.

The truth of this statement is not so obvious as in case of Eq. 9, but an elementary proof, requiring nothing but algebra, is as follows :

* See Salmon : Geometry of Three Dimensions, 4th Edition, p. 535.

† Compare "Ueber der linearen Complexe," . . . p. 37.

‡ It must be borne in mind that spheres cutting the fundamental sphere *from within* at the angle φ , do not belong to the same complex as those which cut it at the same angle *from without*. The same rule applies to internal and external contact.

The coördinates of the two spheres, common to four complexes, being determined by the values of $\xi, \eta \dots$ which satisfy

$$\psi \equiv a\xi + b\eta + c\zeta + d\lambda + e\mu + f\nu = 0$$

$$\psi_1 \equiv a_1\xi + b_1\eta + c_1\zeta + d_1\lambda + e_1\mu + f_1\nu = 0$$

$$\psi_2 \equiv a_2\xi + b_2\eta + c_2\zeta + d_2\lambda + e_2\mu + f_2\nu = 0$$

$$\psi_3 \equiv a_3\xi + b_3\eta + c_3\zeta + d_3\lambda + e_3\mu + f_3\nu = 0$$

and the identical relation

$$II \equiv \xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \mu^2 + \nu^2 \equiv 0,$$

are dependent upon the roots of a quadratic equation. These coördinates can only be real when the discriminant of this quadratic equation is negative.

By putting one coördinate (ν) equal to 1 and solving the first four equations for four coördinates ($\xi, \eta, \zeta, \lambda$) in terms of the other one (μ), then substituting their values in II , the desired quadratic will be obtained. It will be necessary to show that the discriminant of this equation equals D multiplied by a square factor, the form of which depends upon the coördinate retained in II .

Let

$$\Delta \equiv \begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

then

$$\xi = - \frac{\begin{vmatrix} e\mu + f & b & c & d \\ e_1\mu + f_1 & b_1 & c_1 & d_1 \\ e_2\mu + f_2 & b_2 & c_2 & d_2 \\ e_3\mu + f_3 & b_3 & c_3 & d_3 \end{vmatrix}}{\Delta}.$$

This can be expressed by the sum of two determinants, and to avoid complicated terms, it is well to introduce some abbreviations.

Let $\left[\begin{smallmatrix} \Delta \\ a \end{smallmatrix} \right]_e$ be what Δ becomes when a, a_1, a_2, a_3 have been replaced by e, e_1, e_2, e_3 , with similar notation for the other terms, then

$$\Delta^2 \xi^2 = \mu^2 \left[\begin{smallmatrix} \Delta \\ a \end{smallmatrix} \right]_e^2 + 2\mu \left[\begin{smallmatrix} \Delta \\ a \end{smallmatrix} \right]_e \left[\begin{smallmatrix} \Delta \\ a \end{smallmatrix} \right]_f + \left[\begin{smallmatrix} \Delta \\ a \end{smallmatrix} \right]_f^2$$

and similar terms for η, ζ, λ . Substituting them in Π , it becomes

$$\begin{aligned} & \left\{ \left[\frac{D}{a} \right]_e^2 + \left[\frac{D}{b} \right]_e^2 + \left[\frac{D}{c} \right]_e^2 - \left[\frac{D}{d} \right]_e^2 - D^2 \right\} \mu^2 \\ & + 2 \left\{ \left[\frac{D}{a} \right]_e \left[\frac{D}{a} \right]_f + \left[\frac{D}{b} \right]_e \left[\frac{D}{b} \right]_f + \left[\frac{D}{c} \right]_e \left[\frac{D}{c} \right]_f - \left[\frac{D}{d} \right]_e \left[\frac{D}{d} \right]_f \right\} \mu \\ & + \left\{ \left[\frac{D}{a} \right]_f^2 + \left[\frac{D}{b} \right]_f^2 + \left[\frac{D}{c} \right]_f^2 - \left[\frac{D}{d} \right]_f^2 + D^2 \right\} = 0. \end{aligned}$$

The discriminant of this equation can, after a few reductions, be brought to the form

$$\begin{aligned} & \left[\left[\frac{D}{a} \right]_e \left[\frac{D}{b} \right]_f - \left[\frac{D}{a} \right]_f \left[\frac{D}{b} \right]_e \right]^2 + \left[\left[\frac{D}{a} \right]_e \left[\frac{D}{c} \right]_f - \left[\frac{D}{a} \right]_f \left[\frac{D}{c} \right]_e \right]^2 \\ & + \left[\left[\frac{D}{b} \right]_e \left[\frac{D}{c} \right]_f - \left[\frac{D}{b} \right]_f \left[\frac{D}{c} \right]_e \right]^2 - \left[\left[\frac{D}{a} \right]_e \left[\frac{D}{d} \right]_f - \left[\frac{D}{a} \right]_f \left[\frac{D}{d} \right]_e \right]^2 \\ & - \left[\left[\frac{D}{b} \right]_e \left[\frac{D}{d} \right]_f - \left[\frac{D}{b} \right]_f \left[\frac{D}{d} \right]_e \right]^2 - \left[\left[\frac{D}{c} \right]_e \left[\frac{D}{d} \right]_f - \left[\frac{D}{c} \right]_f \left[\frac{D}{d} \right]_e \right]^2 \\ & + D^2 \left[\left[\frac{D}{a} \right]_e^2 - \left[\frac{D}{a} \right]_f^2 + \left[\frac{D}{b} \right]_e^2 - \left[\frac{D}{b} \right]_f^2 + \left[\frac{D}{c} \right]_e^2 - \left[\frac{D}{c} \right]_f^2 - \left[\frac{D}{d} \right]_e^2 + \left[\frac{D}{d} \right]_f^2 - D^2 \right]. \end{aligned}$$

On the other hand, D can be expressed as the product of the two rectangular arrays

$$\begin{Bmatrix} a & b & c & -d & -e & f \\ a_1 & b_1 & c_1 & -d_1 & -e_1 & f_1 \\ a_2 & b_2 & c_2 & -d_2 & -e_2 & f_2 \\ a_3 & b_3 & c_3 & -d_3 & -e_3 & f_3 \end{Bmatrix} \cdot \begin{Bmatrix} a & b & c & d & e & f \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{Bmatrix}$$

which can be developed as the sum of a series of positive and negative squares,

$$\begin{aligned} D \equiv & \left[-D^2 - \left[\frac{D}{d} \right]_e^2 + \left[\frac{D}{d} \right]_f^2 + \left[\frac{D}{c} \right]_e^2 - \left[\frac{D}{c} \right]_f^2 + \left[\frac{D}{b} \right]_e^2 - \left[\frac{D}{b} \right]_f^2 + \left[\frac{D}{a} \right]_e^2 - \left[\frac{D}{a} \right]_f^2 \right] \\ & - (a_1 b_2 e_2 f_3)^2 - (a c_1 e_2 f_3)^2 - (b c_1 e_2 f_3)^2 + (a d_1 e_2 f_3)^2 + (b d_1 e_2 f_3)^2 + (c d_1 e_2 f_3)^2. \end{aligned}$$

The first nine terms are exactly the same as the coefficient of $\mathcal{D}^2$ in the discriminant, and on account of the identity

$$(eb_1c_2d_3)(ab_1c_2f_3) - (fb_1c_2d_3)(ab_1c_2e_3) - (ab_1c_2d_3)(ab_1e_2f_3) = 0^*$$

and similar expressions for the other five terms, the entire discriminant is $\mathcal{D}^2 D$. The negative of this expression is the quantity under the radical sign, in the roots of the quadratic defining μ , and all the other coördinates being linearly expressible in terms of μ , hence; *the spheres common to four linear spherical complexes are real, coincident, or imaginary, according as the combinant of the four complexes is negative, zero, or positive.*†

These properties will now be made use of to classify Dupin's cyclides, according to the arrangement of their nodes and those tangent-planes which are touched by every sphere of the other generation.

At least one of the focal lines must extend to infinity; these points at infinity are the centers of spheres which must touch every sphere belonging to the other generation of the surface, that is, they are planes. With the exception of the case where both focal lines become parabolas, one is an ellipse, the other a hyperbola. Moreover, the direction of the asymptotes indicate the position of the generating planes; in the hyperbola there are two such directions; in the parabola they coincide and in the ellipse become imaginary.

Either generation of the surface may contain points, but not both can contain them; for as every sphere of each generation must touch all the spheres of the other one, if both contained points, the focal lines would have points in common, which is impossible.

The presence of these point-spheres in either generation indicates that each such point is a node of the surface. To apply the criteria obtained above, consider the three complexes defining a cyclide, and the complex of point-spheres, $\lambda = 0$.

The series (12) contains two points, which are real, coincident, or imaginary, according as

$$A = \begin{vmatrix} S_{11} & S_{12} & S_{13} & -d_1 \\ S_{21} & S_{22} & S_{23} & -d_2 \\ S_{31} & S_{32} & S_{33} & -d_3 \\ -d_1 & -d_2 & -d_3 & 1 \end{vmatrix} \quad (17)$$

* See Muir's Determinants, p. 139.

† Evidently this same form (with e, a, f^2 negative, to make all coördinates real, and ν^2 negative in II) holds for two lines common to four linear line complexes. That the lines coincide when D vanishes was known by Plücker (and stated by Klein in his memoir, "Differentialgleichungen der Liniengeometrie," Math. Ann. Vol. 5.)

is negative, zero, or positive. In the same manner, the series contains two planes, which are real, coincident, or imaginary, according as

$$N \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} & -(e_1 + f_1) \\ S_{21} & S_{22} & S_{23} & -(e_2 + f_2) \\ S_{31} & S_{32} & S_{33} & -(e_3 + f_3) \\ -(e_1 + f_1) & -(e_2 + f_2) & -(e_3 + f_3) & 0 \end{vmatrix} \quad (18)$$

is negative, zero, or positive.

Two points are contained among the directrices; they are defined by those values of x, τ which satisfy (13) and

$$d_1 + xd_2 + \tau d_3 = 0. \quad (19)$$

for the radius ρ_κ , of the fundamental circle belonging to the complex ψ_κ is given by the equation

$$\rho_\kappa = \frac{\sqrt{(a_1 + xa_2 + \tau a_3)^2 + (b_1 + xb_2 + \tau b_3)^2 + (c_1 + xc_2 + \tau c_3)^2 - (e_1 + xe_2 + \tau e_3)^2 + (f_1 + xf_2 + \tau f_3)^2}}{(e_1 + xe_2 + \tau e_3) + (f_1 + xf_2 + \tau f_3)} \quad (20)$$

where $\psi_\kappa = \psi_1 + x\psi_2 + \tau\psi_3$. (See Eq. 5.) This circle being a directrix, the complex is a special one, so the numerator can be replaced by $d_1 + xd_2 + \tau d_3$: finally, when this directrix reduces to a point, the radius vanishes, which is expressed in (19).

The values of x, τ common to (13) and (19) are real, coincident, or imaginary, according as

$$M = \begin{vmatrix} S_{11} & S_{12} & S_{13} & d_1 \\ S_{21} & S_{22} & S_{23} & d_2 \\ S_{31} & S_{32} & S_{33} & d_3 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} \quad (21)$$

is positive, zero, or negative.*

Since either generation may contain points, but not both, it follows that *every Dupin's cyclide has four nodes, of which at least two are imaginary, and all may be.*

Two planes are contained among the directrices; they are found by substituting those values of x, τ in (12) that satisfy (13) and

$$e_1 + f_1 + x(e_2 + f_2) + \tau(e_3 + f_3) = 0, \quad (22)$$

* This equation expresses whether the line (19) intersects the conic (13) in real, coincident, or imaginary points in the x, τ plane. See Salmon's Conic Sections, p. 267.

this last equation being analogous to (19), derived from (20) by making the radius of a directrix infinite.

The values of α , τ are real, coincident, or imaginary, according as

$$N \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} & e_1 + f_1 \\ S_{21} & S_{22} & S_{23} & e_2 + f_2 \\ S_{31} & S_{32} & S_{33} & e_3 + f_3 \\ e_1 + f_1 & e_2 + f_2 & e_3 + f_3 & 0 \end{vmatrix} \quad (23)$$

is positive, zero, or negative.

This determinant is identical with (18); when one generation has real planes, those in the other become imaginary except when both pairs become coincident. This is only a corroboration of what was seen before, by geometrical considerations.

It will now be necessary to define the cyclides :

A cyclide having parabolas for focal lines is called a *parabolic cyclide* ; this surface extends to infinity and is of the third degree. It may have no nodes, two nodes, or a double node. In each case two straight lines can be drawn on the surface, at right angles to each other.

All other cyclides are closed surfaces of the fourth degree. There are three general types :

A cyclide having no nodes is called a *ring-cyclide* ; the anchor ring is an illustration.

The nodal cyclides are of two forms, according as the points and planes belong to the same, or different generations ; the former are called *spindle cyclides* ; they consist of two sheets wound about each other, or of one spindle inside another spindle. The latter are called *horn-cyclides* ; they consist of two crescents, with their horns together.

From the physical appearances of these surfaces, there is no difference between series and directrices, so that each must be considered in two ways. For example, the horn-cyclide may have nodes in the first generation (series) and planes in the second (directrices) or conversely. The analytical criteria for the two forms are not the same, and will here be considered independently.

Now, since for $\rho = 0$ and $\rho = \infty$, no attention need be paid to the sign of ρ^2 (spheres with real equations, but imaginary radii), the form of a surface defined by three complexes, is determined by the signs of A , M , N . To assist the mind in forming a picture, the traces of the surfaces on the two planes of symmetry will be added in the following table.

A	M	N	Surface.	Traces on planes of symmetry.	
+	—	—	Ring-cyclide.		
+	—	+			
—	—	+	Horn-cyclide.		
+	+	—			
+	+	+	Spindle-cyclide.		
—	—	—			
0	+	+	Cuspidal-cyclide.		
—	0	—			
0	+	—	Pinch-cyclide.		
—	0	+			
+	—	0	Parabolic Ring-cyclide.		
+	—	0			
+	+	0	Parabolic Binoidal-cyclide.		
—	—	0			
0	+	0	Parabolic Cuspidal-cyclide.		
—	0	0			

This completes the solution of the general problem.

Cyclides of revolution have a circle for one focal line, and a straight line from the other. They must be either ring-cyclides or spindle-cyclides, as the spheres whose centers lie on the circle must all be of the same radius.

If this be the series, the sheaf (12) can be reduced to the form

$$\phi_1 + x\phi_2 + \tau\phi_3 = m\lambda + n(\mu - \nu)$$

for some values of x, τ . The determinants in the coefficients, which vanish to satisfy this equation, give the necessary and sufficient condition that the series is one of rotation.

If the second generation has a circle for a focal-line, all the directrices must have the same radius ρ , so

$$\rho = \frac{d_1 + xd_2 + \tau d_3}{e_1 + f_1 + x(e_2 + f_2) + \tau(e_3 + f_3)}$$

must be independent of the value of x , τ , or

$$d_1 : d_2 : d_3 = (e_1 + f_1) : (e_2 + f_2) : (e_3 + f_3).$$

The *cone* is a cyclide having one generation composed of planes ; ($m = 0$ in the above form) or

$$e_1 + f_1 = e_2 + f_2 = e_3 + f_3 = 0.$$

The *circle* is a cyclide having one generation composed of points.

$$n = 0, \text{ or } d_1 = d_2 = d_3 = 0.$$

The *right line* is a cyclide, with one generation of points and one of planes. Combination of above criteria. These forms were treated in my paper named above.

CORNELL UNIVERSITY, *Aug.*, 1896.